# Boundary-Marching Method for Discontinuity Analysis in Waveguides of Arbitrary Cross Section

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*Abstract*—**A recursive algorithm previously used in diffusion problems of geophysics and in electrostatics, is extended to wave phenomena. It is used to construct a matrix representation for an infinitely long waveguide of arbitrary cross-sectional shape. This representation is used in finite element analysis of waveguide discontinuities. In numerical tests, scattering matrices for the long guides converge to nearly full word-length in 6–7 recursion steps, and discontinuity characteristics are within 1%–2% of known results where they exist.**

## I. Introduction

FINITE elements formulated in terms of vector field components have been widely used in characterizing arbitrarily shaped waveguides. Recently, the finite element method has also been successfully applied in analyzing some subclasses of waveguide discontinuity problems which are essentially two-dimensional, such as E-Plane and H-plane discontinuities. However, a general waveguide discontinuity is three-dimensional. It joins two or more waveguides, possibly dissimilar, it has an arbitrary shape in all directions, and may contain inhomogeneous materials. In such cases, a full three dimensional analysis is required. When using the finite element method, one also needs to model the two infinite waveguides directly attached to the discontinuity section. The most common way of dealing with such infinite guides is to truncate the guides at a distance sufficiently far from the discontinuity, with a large number of mesh nodes. Proper boundary conditions are then applied at the truncated surfaces, assuming that the field decays significantly before reaching the truncations. This approach results in an undesirably large mesh and therefore is not practical in many three-dimensional problems. For some special cases, where the waveguide geometry is rectangular, circular or elliptical, an equivalent boundary condition, derived from analytically known guided modes, may be used to truncate the mesh at a smaller distance from the discontinuity. However, if the geometry of the waveguide structure is more general, there is no analytic solution for the waveguide and an equivalent boundary condition for the infinite guide section cannot be found easily. This is especially true in the case of inhomogeneously dielectric-loaded waveguides.

This paper presents a very general finite element scheme which can be used to model an arbitrarily-shaped guide that may be inhomogeneous in the transverse direction. The algorithm uses a simple recursive method to generate a submatrix which relates the field characteristics on the near-field surface to the field conditions on the far-field surface. As a result, it can be used to truncate the finite element mesh at a distance very close to the discontinuity without losing any generality, for any arbitrarily-shaped guide. This procedure resembles the "roof-raising" process used in static and diffusion fields by Kisak, Silvester and Telford [1], and the related but more general two-dimensional "ballooning" algorithm applied to electrostatics problems by Silvester, Lowther, Carpenter and Wyatt [2].

Waveguide analysis with the finite element method has long been troubled by the appearance of spurious modes. Although these are commonly encountered in eigenvalue problems, it has been shown that spurious modes can affect solutions to deterministic problems also [3], [4]. The orthospectral ("spectrally correct") elements obtained by using mixed-order approximating functions on hexahedra [5], [6] have been shown to produce solutions free of spurious modes. All the results reported in this paper are based on elements of this type and, as would be expected, no spurious-mode corruption of solutions has been encountered.

## II. Variational Formulation

The general configuration of the class of problems considered here is illustrated in Fig. 1. The waveguide discontinuity or junction region is viewed as being composed of three subregions: (*i*) a uniform guide $\Omega_1$, (*ii*) the discontinuity region proper, $\Omega_d$, and (*iii*) the second uniform guide $\Omega_2$, not necessarily similar to $\Omega_1$. In the uniform guides and in the discontinuity region $\Omega_d$ the electric field must everywhere satisfy the vector Helmholtz equation:

$$\frac{1}{\mu_r} \nabla \times \nabla \times \boldsymbol{E} - k_0^2 \epsilon_r \boldsymbol{E} = 0, \tag{1}$$

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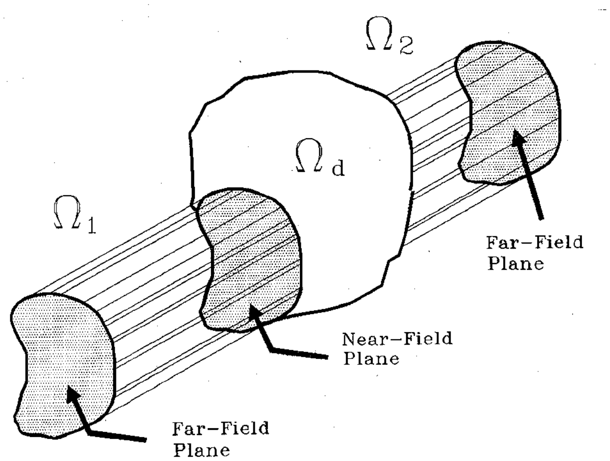


Fig. 1. The three subregions of a waveguide junction or discontinuity problem: the discontinuity $\Omega_d$ proper, flanked by two infinite waveguides $\Omega_1$ and $\Omega_2$.

subject to boundary conditions of the following types:

$$(\nabla \times \boldsymbol{E}) \times \mathbf{1}_n = 0 \quad \text{(homogeneous Neumann on magnetic wall)}, \tag{2}$$

$$\boldsymbol{E} \times \mathbf{1}_n = 0 \quad \text{(homogeneous Dirichlet on perfect electric conductors)}, \tag{3}$$

$$\boldsymbol{E} \times \mathbf{1}_n = \boldsymbol{E}_0 \quad \text{(inhomogeneous Dirichlet on excitation planes)}. \tag{4}$$

As is well known [7], solving the Helmholtz equation for the electric field vector in the lossless case is equivalent to extremizing the variational functional

$$\mathfrak{F}(\boldsymbol{E}) = \frac{1}{2} \int_\Omega \left\{ -\frac{1}{\mu_r} (\nabla \times \boldsymbol{E}) \cdot (\nabla \times \boldsymbol{E}) + k_0^2 \epsilon_r \boldsymbol{E} \cdot \boldsymbol{E} \right\} d\Omega. \tag{5}$$

The finite element methods contemplated here rely on this formulation, and the boundary-marching method fits into the general framework of variational methods very well. The results given by Marcuvitz [8] are quite extensive enough to serve for independent verification of all calculations.

## III. Discretization

In order to avoid the appearance of spurious modes, orthospectral (mixed-order hexahedral) finite elements cast in terms of the projection components [5] are utilized. That is, the electric field vector $\boldsymbol{E}$ is written

$$\boldsymbol{E} = 1_\xi E_\xi + 1_\eta E_\eta + 1_\nu E_\nu, \tag{6}$$

where $1_\xi$, $1_\eta$, $1_\nu$ are reciprocal unitaries of the local coordinates, and $E_\xi$, $E_\eta$, $E_\nu$ are the covariant projection components of $\boldsymbol{E}$. Each component of the electric field $\boldsymbol{E}$ in each element is approximated by element functions $\alpha_m^\xi(\xi, \eta, \nu)$, $\alpha_m^\eta(\xi, \eta, \nu)$, $\alpha_m^\nu(\xi, \eta, \nu)$:

$$E_\xi = \sum_{m=1}^{18} E_m^\xi \alpha_m^\xi(\xi, \eta, \nu), \tag{7}$$

$$E\eta = \sum_{m=19}^{36} E_m^\eta \alpha_m^\eta(\xi, \eta, \nu), \tag{8}$$

$$E_\nu = \sum_{m=37}^{54} E_m^\nu \alpha_m^\nu(\xi, \eta, \nu). \tag{9}$$

Discretizing the entire region into finite elements and applying the standard finite element minimization procedure to the functional [7], [9] gives the following system of equations:

$$\left\{ -\frac{1}{\mu_r} [S] + k_0^2 \epsilon_r [T] \right\} [E] = [0]. \tag{10}$$

Here $[E]$ is a column matrix representing the nodal electric fields, $[S]$ is the square matrix that results from the curl-curl term in the functional, and $[T]$ corresponds to the dot-product term in the functional.

## IV. Boundary-Marching Algorithm

Four steps of the boundary-marching process are illustrated in Fig. 2. As indicated, the far-field and near-field planes are initially placed at the same location; then the far-field plane is moved away step by step, with the distance at each step growing larger as the far-field plane recedes. To develop the computational algorithm, let $\Omega^0$ be the volume of the initial segment of a uniform guide; let $\Gamma_1$ and $\Gamma_2$ be the two surfaces enclosing this segment of the guide. Because the guide is uniform, $\Gamma_1$ and $\Gamma_2$ are congruent. The guide segment is discretized into finite elements, with the following restriction on the manner of subdivision: the placement of finite element nodes and edges must leave $\Gamma_1$ and $\Gamma_2$ congruent, i.e., the element and node placement on $\Gamma_1$ must correspond exactly to that on $\Gamma_2$. On discretizing the segment into a number of finite elements and minimizing the corresponding electric-field functional, the following set of simultaneous equations is obtained:

$$\begin{bmatrix} [W]_{11} & [W]_{1i} & [W]_{12} \\ [W]_{i1} & [W]_{ii} & [W]_{i2} \\ [W]_{21} & [W]_{2i} & [W]_{22} \end{bmatrix} \begin{bmatrix} [E]_1 \\ [E]_i^0 \\ [E]_2 \end{bmatrix} = 0. \tag{11}$$

The matrix equation had been partitioned so that the submatrices identified by subscript 1 correspond to finite element nodes located on plane 1 (the near-field plane), those identified by subscript 2 correspond to finite element nodes located on plane 2, and those identified by $i$ are in the interior of the guide segment. (The superscripts are of no significance for the moment; they are introduced only for notational consistency with further development.) Thus $[E]_1$ represents the nodal electric fields on boundary surface 1, $[E]_2$ the nodal electric fields on boundary plane 2,

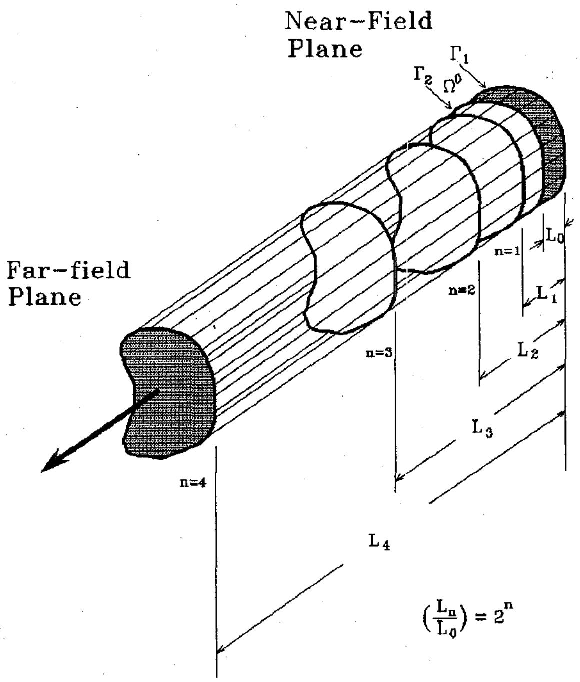


Fig. 2. The boundary marching process. The finite guide length $L_n$ doubles at every step, reaching full floating-point precision after 5 or 10 steps.

and $[E]_i^0$ is the set of all nodal electric fields inside the region. The internal field nodes $[E]_i^0$ are not of interest, they are eliminated by a static condensation scheme [9], [10]. The system of equations for waveguide segment $\Omega^0$ is thereby reduced to the more compact form

$$\begin{bmatrix} [S]_{11}^0 & [S]_{12}^0 \\ [S]_{21}^0 & [S]_{22}^0 \end{bmatrix} \begin{bmatrix} [E]_1^0 \\ [E]_2^0 \end{bmatrix} = 0, \qquad (12)$$

where the number of matrix rows and columns equals the number of nodal variables on the bounding surfaces $\Gamma_1$ and $\Gamma_2$ only; the subscript $i$ has disappeared altogether. The new submatrices $[S]_{mn}^0$ are obtained from the nine submatrices $[W]_{mn}$ of the finite element functional by

$$[S]_{11}^0 = [W]_{11} - [W]_{1i}([W]_{ii})^{-1}[W]_{i1} \qquad (13)$$

$$[S]_{12}^0 = [W]_{12} - [W]_{1i}([W]_{ii})^{-1}[W]_{i2} \qquad (14)$$

$$[S]_{21}^0 = [W]_{21} - [W]_{2i}([W]_{ii})^{-1}[W]_{i1} \qquad (15)$$

$$[S]_{22}^0 = [W]_{22} - [W]_{2i}([W]_{ii})^{-1}[W]_{i2}. \qquad (16)$$

The notation follows that for the full matrix representation: submatrix $[S]_{mn}^0$ interrelates field components associated with nodes on surface $\Gamma_m$ of the condensed element with those on surface $\Gamma_n$.

Now the coefficient matrix of (9) describes the interrelationship of electric field components at the two ends of a fixed length of uniform guide, with no assumptions as to the length (it need not be smaller than a wavelength). A section of guide twice that length can therefore be modeled without loss of accuracy by cascading two such matrices. The description of a double-length segment of guide is thus obtained by combining two identical segments as described by (9) and enforcing the field continuity on their common boundary:

$$\begin{bmatrix} [S]_{11}^0 & [S]_{12}^0 & 0 \\ [S]_{21}^0 & [S]_{11}^0 + [S]_{22}^0 & [S]_{12}^0 \\ 0 & [S]_{21}^0 & [S]_{22}^0 \end{bmatrix} \begin{bmatrix} [E]_1^0 \\ [E]_i^0 \\ [E]_2^0 \end{bmatrix} = 0. \qquad (17)$$

As before, the electric field at the internal nodes is of no interest. It can be eliminated from further consideration, removing $[E]_i^0$ by the same process of static condensation as previously. What results is a description of the "super-element" twice as long as the original one:

$$\begin{bmatrix} [S]_{11}^1 & [S]_{12}^1 \\ [S]_{21}^1 & [S]_{22}^1 \end{bmatrix} \begin{bmatrix} [E]_1^1 \\ [E]_i^1 \end{bmatrix} = 0, \qquad (18)$$

where the superscript 1 indicates that one doubling of the guide length has taken place. The four new submatrices are obtained much as before:

$$[S]_{11}^1 = [S]_{11}^0 - [S]_{1i}^0([S]_{ii}^0)^{-1}[S]_{i1}^0, \qquad (19)$$

$$[S]_{12}^1 = -[S]_{1i}^0([S]_{ii}^0)^{-1}[S]_{i2}^0, \qquad (20)$$

$$[S]_{21}^1 = -[S]_{2i}^0([S]_{ii}^0)^{-1}[S]_{i1}^0, \qquad (21)$$

$$[S]_{22}^1 = [S]_{22}^0 - [S]_{2i}^0([S]_{ii}^0)^{-1}[S]_{i2}^0. \qquad (22)$$

The new super-element representing the guide section is twice as long, and therefore has twice the volume of the original section: $\Omega^1 = 2\Omega^0$.

Further lengthening of the guide segment is achieved by applying the same procedure recursively. After each of $k$ recursions, the matrix relating the excitation field on boundary surface 1 to the field distribution on the far-field plane can be expressed in terms of the submatrices of the previous recursion as follows:

$$[S]_{11}^{k+1} = [S]_{11}^k - [S]_{1i}^k([S]_{ii}^k)^{-1}[S]_{i1}^k \qquad (22)$$

$$[S]_{12}^{k+1} = -[S]_{1i}^k([S]_{ii}^k)^{-1}[S]_{i2}^k \qquad (23)$$

$$[S]_{21}^{k+1} = -[S]_{2i}^k([S]_{ii}^k)^{-1}[S]_{i1}^k \qquad (24)$$

$$[S]_{22}^{k+1} = [S]_{22}^k - [S]_{2i}^k([S]_{ii}^k)^{-1}[S]_{i2}^k. \qquad (25)$$

At each recursion step, the length of the super-element (i.e., of the guide segment) is augmented by a factor of 2. Consequently, in the course of $N$ recursions, the length of the uniform guide in the propagation direction grows by a factor of $2^N$. In effect the procedure is equivalent to marching out the boundary of the uniform guide from the excitation plane to the far-field plane, sufficiently far for all evanescent modes to decay to a negligible level. The method therefore provides a simple way for simulation and investigation of wave propagation in any arbitrarily-shaped guide. The advantage of the algorithm becomes even more pronounced in the case of inhomogeneous dielectric-loaded guides, where any competitive method currently available (e.g., any integral equation method) becomes too complicated, if not impossible.

To summarize the full computational algorithm: The finite element matrix for a finite, generally quite short, length of guide is constructed using standard techniques. It is partitioned into submatrices according to whether the node numbers refer to bounding plane 1 (the near-field plane), plane 2 (the farther plane), or the interior $i$. The nine submatrices are then manipulated in an $N$-step recursion as follows.

*Step 1: Initialize.*

$$[S]_{11}^0 = [W]_{11} - [W]_{1i}([W]_{ii})^{-1}[W]_{i1}$$

$$[S]_{12}^0 = [W]_{12} - [W]_{1i}([W]_{ii})^{-1}[W]_{i2}$$

$$[S]_{21}^0 = [W]_{21} - [W]_{2i}([W]_{ii})^{-1}[W]_{i1}$$

$$[S]_{22}^0 = [W]_{22} - [W]_{2i}([W]_{ii})^{-1}[W]_{i2}.$$

*Step 2: March out, for $k = 0, \cdots, N-1$.*

$$[S]_{11}^{k+1} = [S]_{11}^k - [S]_{1i}^k([S]_{ii}^k)^{-1}[S]_{i1}^k$$

$$[S]_{12}^{k+1} = -[S]_{1i}^k([S]_{ii}^k)^{-1}[S]_{i2}^k$$

$$[S]_{21}^{k+1} = -[S]_{2i}^k([S]_{ii}^k)^{-1}[S]_{i1}^k$$

$$[S]_{22}^{k+1} = [S]_{22}^k - [S]_{2i}^k([S]_{ii}^k)^{-1}[S]_{i2}^k$$

*Step 3: Record result.*

$$\begin{bmatrix} [S]_{11}^N & [S]_{12}^N \\ [S]_{21}^N & [S]_{22}^N \end{bmatrix} \begin{bmatrix} [E]_1^N \\ [E]_i^N \end{bmatrix} = 0. \tag{27}$$

The guide length grows as $2^N$, so only 20 recursion steps will turn an initial explicit guide model millimeters long into the same number of kilometers.

## V. Numerical Results

The boundary-marching technique has been extensively tested on guides of several different cross-sectional shapes. All test programs used Crowley-type orthospectral elements [5] and were written in the Ada language. Two computers were used: an 80386-based (MS-DOS) machine for program development and debugging, followed by a Cray X-MP supercomputer for subsequent production runs. The algorithm is first tested here by investigating the decay behavior of the evanescent modes, as the boundary of the far-field plane is marched out from the excitation plane. Fig. 3 shows how the reflected waves of the $TE_{0,1}$, $TE_{0,2}$ and $TE_{2,0}$ modes decay with guide length in a typical rectangular waveguide. All computations were carried out in 64-bit arithmetic, so that noticeable roundoff error accumulation in the fourteenth or fifteenth digit is to be expected. In fact, the roundoff error falls substantially below that level in two of the modes. The magnitude of the forward transmission of the scattering coefficient of the propagating mode ($TE_{1,0}$ mode), as a function of the length of the waveguide is also shown in Fig. 3. Clearly, about 6 or 7 recursions are more than adequate in this case to make the guide "infinite" for all practical purposes. Even quite near cutoff, about 20 or 30 recursion steps suffice, yielding a waveguide length of $10^5$–$10^8$ times guide width.

To illustrate how the boundary-marching procedure can

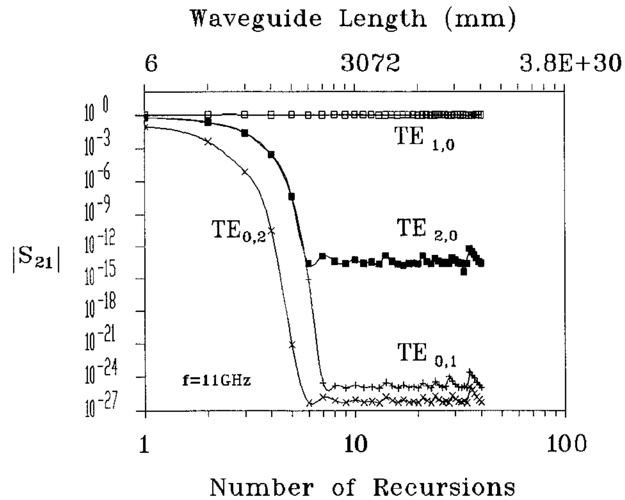


Fig. 3. Scattering of dominant and evanescent modes in a rectangular guide of 20.32 mm × 10.16 mm cross-section, as a function of the number of recursions or equivalent guide length.

be incorporated into waveguide analysis, Fig. 4 shows the finite element scheme for the solution of a zero-thickness capacitive window. This problem is well established, and results may be found in Marcuvitz [8]. The reference plane of the transmission and reflection parameters is calibrated by using a matched "through" segment. The calibration also eliminates any error that may arise as a result of signal loss or phase distortions in the waveguide segments. The scattering parameters of the structure are first computed with the discontinuity section replaced by a segment of empty waveguide of known finite length. The calibration factors are determined by setting the magnitude of the forward transmission of the scattering coefficients to unity and the phase to 90 degrees. Then the scattering parameters of the structure including the discontinuity are computed by replacing the "calibration segment" with the discontinuity. Dividing the transmission and reflection parameters by the proper calibration factor, and subtracting the phase offset from the phase angles of the parameters, one obtains the final scattering parameters at the desired reference plane. By using the same segment length for the calibration segment as for the discontinuity segment, the transmission and reflection parameters are calibrated to the plane where the zero-thickness window resides. If desired, the parameters could also be calibrated to some other reference plane, by choosing different lengths for the calibration segment and the discontinuity segment.

Phase and amplitude of the forward transmission coefficient of the scattering parameters of the capacitive window are shown in Fig. 5, for different frequencies. The results agree with the analytical approximation given by Marcuvitz [8] to within 1–2 percent. Since the Marcuvitz solution is not exact, it cannot be used to establish firm error bounds. However, the finite element solution and Marcuvitz's approximation are thought to incur errors of roughly similar magnitude, so their agreement is held to confirm the validity of the boundary-marching technique.

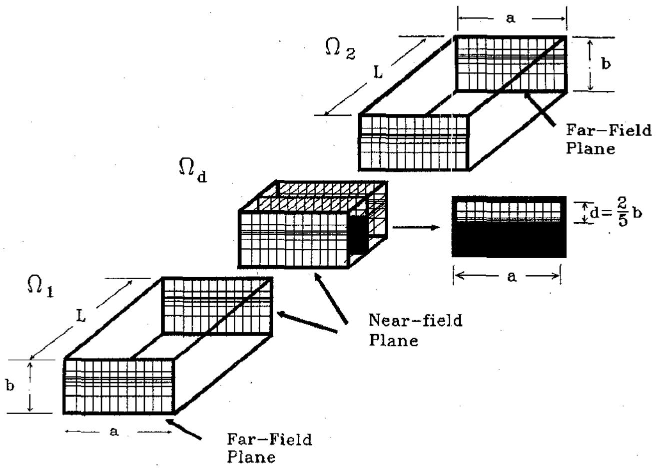


Fig. 4. The classical zero-thickness capacitive window problem, modeled as the three regions of Fig. 1.

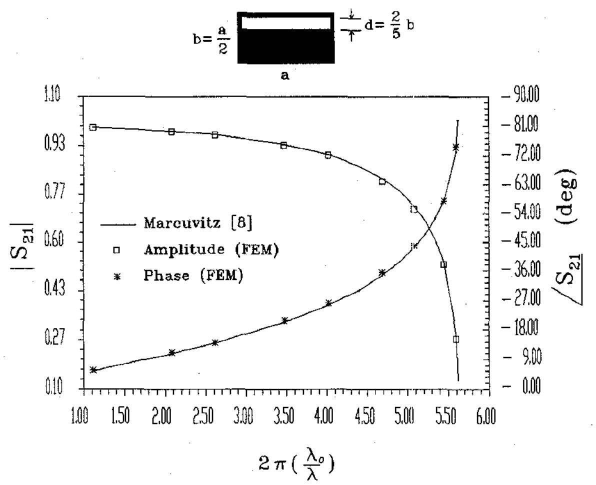


Fig. 5. Forward transfer scattering parameter for the capacitive window problem: comparison of curves given by Marcuvitz (solid line) with finite element computations.

## VI. Conclusions

A general recursive method has been proposed and validated for creating finite element models of very great lengths (thousands of free-space wavelengths) of arbitrarily-shaped waveguide. The method is valid for any guide, so long as a technique is available for constructing a finite element model of a finite length of the guide. It is particularly useful for analysis of waveguide components and discontinuity regions, where it permits truncation of the finite element mesh very close to the discontinuity region without compromising result accuracy. It does not introduce any error beyond the discretization error inherited from the finite element meshing; it is unconditionally stable, except possibly at frequencies very close to the cutoff frequency of the lowest eigenmode, where the ratio of free-space wavelength to guided wavelength approaches the floating-point precision available. This algorithm appears to be particularly useful for discontinuity analyses involving inhomogeneous dielectric-loaded guides, such as finline and shielded microstrips, but further verifying work is needed to establish what limits there may be to its use.

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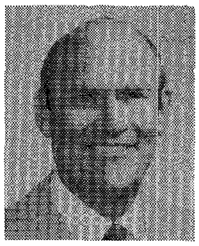

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